Color Filter Design For Multiple Illuminants and Detectors

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Abstract

This paper presents one approach to designing color filters for a colorimeter that uses multiple internal illuminants and multiple filtered-detectors. The internal illuminants and optical detectors are fixed items in the colorimeter. The color filters are designed using simplex search, Vora's measure of goodness, and linear minimum mean square estimation. Radiance and reflectance data sets were used to test the performance of the designed color filters. Design experiments for solely photometric, solely radiometric, and combination colorimeters gave average ΔE_{Lab} errors below 0.6, 2.3, and 1.5 respectively.

1 Introduction and Problem Description

The standard colorimeter described in [10, 2, 5] uses one illuminant, one detector, and three color filters. The color filters are designed to be an approximation to a linear combination of the CIE color matching functions. From three measurements, estimates of CIE XYZ and CIELab values are made. Errors in the filter approximations to the CIE color matching functions will result in errors in the color values.

The colorimeter proposed in this paper will use multiple illuminants, detectors, and color filters to improve estimates of tristimulus values. By making measurements with different combinations of internal illuminants, optical detectors, and correctly designed color filters, the number of useful measurements and the range space of the measurement are increased, and the sensitivity to noise and filter errors are decreased. If the human visual subspace (HVSS) [5] is nearly contained in the range space of the measurements, then accurate estimates of CIE tristimulus values can be found from the measured data using linear minimum mean squared (LMMSE) estimation matrices [1].

This work describes a method of designing color filters that maximizes the distance between the HVSS and the range space of the measurements. In Section 2, mathematical notation and photometric and radiometric measurement matrices are introduced. The simplex search method is used to maximize a performance function based on Vora's measure of goodness [6] with respect to the unknown transmittance spectra of the color filters. Section 3 describes the design experiments carried out for task specific colorimeters. One color filter set was designed for a photometric colorimeter; another for a radiometric colorimeter; and the last for a combination colorimeter. All design experiments were performed using sampled data, and all equations are expressed using vector space notation [5].

2 Approach

The CIE tristimulus values are defined in matrix/vector notation by $\mathbf{t} = \mathbf{A}^T\mathbf{L}\mathbf{r}$ where $\mathbf{A}$ is the $N \times 3$ color matching matrix of CIE color matching functions, $\mathbf{L}$ is the $N \times N$ diagonal matrix for the viewing illuminant with $N \times 1$ radiance spectrum $\mathbf{l}$, and $\mathbf{r}$ is the $N \times 1$ reflectance spectrum of the object being viewed. The color matching matrix under illuminant $\mathbf{L}$ is defined as $\mathbf{A_L} = \mathbf{LA}$ [5]. Thus the tristimulus values are also defined as $\mathbf{t} = \mathbf{A_L}^T\mathbf{r}$.

The measurement for a single internal illuminant is given by $\mathbf{c} = \mathbf{M}^T\mathbf{D}\mathbf{L_s}\mathbf{r}$ where $\mathbf{M}$ is the $N \times N_f$ matrix of N_f color filter transmittance spectra, $\mathbf{D}$ is the $N \times N$ diagonal matrix of sensitivity spectrum $\mathbf{d}$ for the detector, and $\mathbf{L_s}$ is the $N \times N$ diagonal matrix of radiance spectrum $\mathbf{l_s}$ for the internal illuminant [5]. The latter equation needs to be modified for the multiple illuminant, multiple filtered-detector measurement device. All data were sampled for a visible wavelength range of 390-730nm at 2nm intervals; thus $N = 171$.

The radiance and the sensitivity spectra of the illuminants and detectors used in Section 3 are known. In Section 3, the radiance spectra of the LED illuminants and the sensitivity spectra of the detectors were specified by the Color Savvy company. Thus the photometric and radiometric measures can be written as "measurement matrices" that are matrix functions of the unknown transmittance spectra of the color filters and known radiance and the sensitivity spectra of the illuminants and detectors.

Suppose there are N_f unknown color filters, N_d known optical detectors and N_l known internal illuminants. The colorimeter makes $K = N_f N_d N_l$ photometric measurements and $P = N_f N_d$ radiometric measurements. If the detectors have spectral sensitivities $\mathbf{d}_i$, the internal illuminants have radiance spectra $\mathbf{l}_{\mathbf{s}_j}$, and color filter transmittance spectra $\mathbf{m}_k$, then the photometric measurement matrix which characterizes the K spectral sensitivities may be written as

$$\begin{aligned}\mathbf{V}(\mathbf{M}) = \; & [\mathbf{D}_1[\mathbf{L}_{\mathbf{s}1}\mathbf{M}, \mathbf{L}_{\mathbf{s}2}\mathbf{M}, ..., \mathbf{L}_{\mathbf{s}N_l}\mathbf{M}], \mathbf{D}_2[\mathbf{L}_{\mathbf{s}1}\mathbf{M}, \mathbf{L}_{\mathbf{s}2}\mathbf{M}, ..., \mathbf{L}_{\mathbf{s}N_l}\mathbf{M}], ... \\ & \mathbf{D}_{N_d}[\mathbf{L}_{\mathbf{s}1}\mathbf{M}, \mathbf{L}_{\mathbf{s}2}\mathbf{M}, ..., \mathbf{L}_{\mathbf{s}N_l}\mathbf{M}]]\end{aligned} \tag{1}$$

where $\mathbf{D}_i = diag(\mathbf{d}_i)$, $\mathbf{L}_{\mathbf{s}j} = diag(\mathbf{l}_{\mathbf{s}j})$, and $\mathbf{M} = [\mathbf{m}_1, \mathbf{m}_2, ..., \mathbf{m}_{N_f}]$. Similarly, the radiometric measurement matrix that characterizes the P sensitivities is given by

$$\mathbf{W}(\mathbf{M}) = \; [\mathbf{D}_1\mathbf{M}, \mathbf{D}_2\mathbf{M}, ..., \mathbf{D}_{N_d}\mathbf{M}]. \tag{2}$$

The transmittance spectra of the optimal color filter set are parameterized to reduce the number of variables needed to specify $\mathbf{M}$ and to satisfy certain physical constraints. The transmittance spectra of physically realizable color filters must be non-negative [4]; in addition, the thin-film filter manufacturing process requires a smooth theoretical color filter transmittance spectrum [8]. In the design experiments, transmittance spectra of the color filter set were constrained to be Gaussian or sum of Gaussians shaped curves:

$$m_i(x) = \rho_i \frac{1}{\sqrt{2\pi}\sigma_{i1}} exp[-\frac{(x-\mu_{i1})^2}{2\sigma_{i1}^2} + (1-\rho_i)\frac{1}{\sqrt{2\pi}\sigma_{i2}} exp[-\frac{(x-\mu_{i2})^2}{2\sigma_{i2}^2}] \tag{3}$$

where μ_{i1}, μ_{i2}, σ_{i1}^2 and σ_{i2}^2 are unknown variables, ρ_i is an unknown variable between 0 and 1, and x is a wavelength in the range of 390-730 nm. The unknown transmittance color filter spectrum may be parameterized as a vector

$$\mathbf{v} = [\mu_{11}, \sigma_{11}^2, .., \mu_{N_f1}, \sigma_{N_f1}^2, \rho_1, \mu_{12}, \sigma_{12}^2, .., \rho_{N_f}, \mu_{N_f2}, \sigma_{N_f2}^2]. \tag{4}$$

Thus the number of variables needed to characterize $\mathbf{M}$ was reduced from $N \times N_f$ to $5 \times N_f$.

The optimal filter set is found by maximizing Vora's measures of goodness[6]. The mathematical description of Vora's measure of goodness ν is given in Appendix A. An advantage of this measure is that its analytic form works well with most optimization algorithms. If the reflectance source is illuminated by a viewing illuminant with a radiance spectrum $\mathbf{l}$, then the photometric measure of goodness is $\nu(\mathbf{A_L}, \mathbf{V})$. The combined measure of goodness used to design the color filters is given by

$$C(\mathbf{V}, \mathbf{W}) = [(\gamma)\nu(\mathbf{A_L}, \mathbf{V}(\mathbf{M})) + (1-\gamma)\nu(\mathbf{A}, \mathbf{W}(\mathbf{M}))]. \tag{5}$$

where γ is the weight between 0 and 1 which specifies the importance for photometric measure $\nu(\mathbf{A_L}, \mathbf{V}(\mathbf{M}))$ over radiometric measure $\nu(\mathbf{A}, \mathbf{W}(\mathbf{M}))$. Different photometric color filter sets were not designed for each viewing illuminant; rather, one color filter set was designed for the CIE A viewing illuminant and used for measurements under all other viewing illuminants.

The simplex search method [9] is applied to Eq(5) to find the optimal parameter vector $\mathbf{v}$ from Eq(4). The **fmins** function from MATLAB uses the simplex search on $1 - C(\mathbf{V}, \mathbf{W})$. The photometric error, $e_p = E\{||\mathbf{A_L}^T\mathbf{r} - \mathbf{R}\mathbf{V}^T\mathbf{r}||\}$, is minimized when

$$\mathbf{R} = (\mathbf{A_L}^T E[\mathbf{r}\mathbf{r}^T]\mathbf{V})(\mathbf{V}^T E[\mathbf{r}\mathbf{r}^T]\mathbf{V})^{-1} \tag{6}$$

where $E\{\cdot\}$ is the expected value operator. $\mathbf{R}$ is called the photometric LMMSE matrix. Likewise, the radiometric LMMSE matrix,

$$\mathbf{S} = (\mathbf{A}^T E[\mathbf{g}\mathbf{g}^T]\mathbf{W})(\mathbf{W}^T E[\mathbf{g}\mathbf{g}^T]\mathbf{W})^{-1}, \tag{7}$$

minimizes the radiometric error $e_r = E\{||\mathbf{A}\mathbf{g} - \mathbf{S}\mathbf{W}^T\mathbf{g}||\}$. $E[\mathbf{g}\mathbf{g}^T]$ and $E[\mathbf{r}\mathbf{r}^T]$ are the correlation matrices of radiance and reflectance spectra respectively.

3 Design Experiments and Results

In the first experiment, a photometric colorimeter was designed, i.e. $\gamma = 1$ in Eq(5). Five LEDs, whose radiance spectra are provided by Color Savvy and shown in Figure 1, were used as the internal illuminant set. The optical detector was assumed to be uniform, making $\mathbf{d}$ a vector of 1's. The number of color filters was four. The transmittance spectra of the color filters were constrained to be Gaussian shaped curves, making $\rho_i = 1$ for $i = 1, ..., 4$. The correlation matrix $E[\mathbf{r}\mathbf{r}^T]$ was obtained from the Dupont paint sample set [7]. The optimal color filters are shown in Figure 2. The goodness of the photometric colorimeter versus all pertinent HVSS is shown in Table 1. The colorimetric quality factor τ defined in Appendix B is included in the goodness tables because it is a measure frequently used in color science. The experimental results on reflective sets described in [1] of photometric measurements made under different CIE standard illuminants are shown in Table 2. The worst case reflectance measurements are made under the fluorescent viewing illuminant due to the spikes in F2 spectrum [3].

A radiometric colorimeter, where $\gamma = 1$ in Eq(5), was designed in the second experiment using a uniform detector and four Gaussian-shaped filters. The illuminant and reflectance set used to generate the radiance spectra set for the computation of $E[\mathbf{g}\mathbf{g}^T]$ are listed in the first column of Table 3. The radiometric colorimeter based on these filters was tested

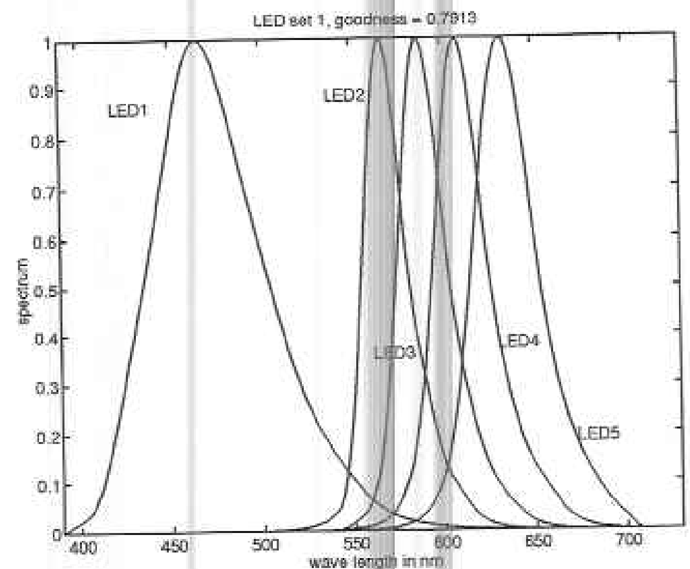

Figure 1: The radiance spectra of the LEDs used for the internal illuminant sources were provided by the Color Savvy company.

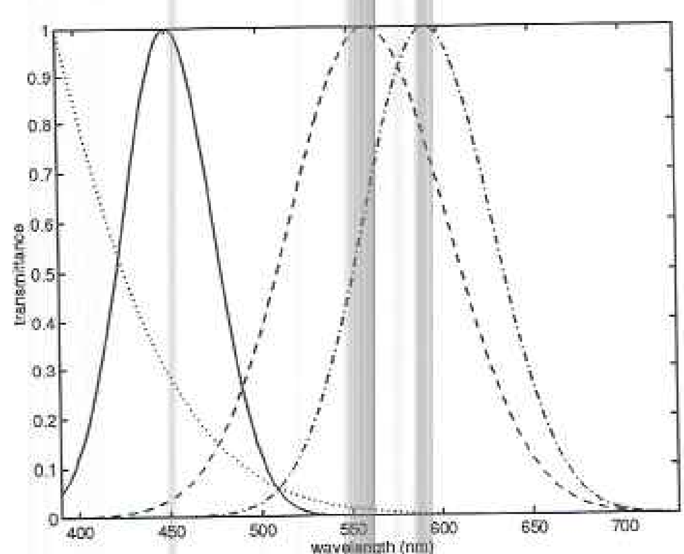

Figure 2: These are the optimal photometric Gaussian-shaped color filters for the Color Savvy LED set.

color matching function	ν	τ
$\mathbf{A}$	0.9976	0.7244
$\mathbf{A}_A^*$	0.9995	0.6678
$\mathbf{A}_{D65}$	0.9997	0.7263
$\mathbf{A}_F$	0.8477	0.4495

Table 1: This table lists the goodness for the measurement matrix $\mathbf{V}$ versus the color matching function in the first column. The * denotes the color matching function that the color filters were designed for. The color filters were constrained Gaussian shaped transmittance spectra. $\gamma = 1$.

Spec. ⇓	viewing illum.	ϵ_{mean}	ϵ_{max}	ΔE_{mean}	ΔE_{max}
Munsell	A	0.0005	0.0027	0.0351	0.1721
Dupont	A	0.0007	0.0024	0.0381	0.1815
Litho	A	0.0005	0.0026	0.0286	0.0637
Object	A	0.0007	0.0053	0.0789	0.4651
Munsell	D65	0.0021	0.0088	0.0969	0.3609
Dupont	D65	0.0025	0.0081	0.1701	0.8286
Litho	D65	0.0021	0.0104	0.0917	0.2294
Object	D65	0.0016	0.0133	0.1180	1.0138
Munsell	F2	0.0051	0.0235	0.2265	0.7287
Dupont	F2	0.0015	0.0121	0.1213	0.4056
Litho	F2	0.0237	0.0682	0.5737	1.0727
Object	F2	0.0172	0.8157	0.3496	2.3322

Table 2: An optimal Gaussian-shaped color filter set, designed under viewing illuminant A with $\gamma = 1$, is tested for its color measurement accuracy using both reflectance sets and radiance sets. The five Color Savvy LEDs were used as the internal illuminant.

Radiometric device, $\nu = 0.9975$, $\tau = 0.9956$						
Expected Spec.	Rad. Spec. ⇓	orig. source	ϵ_{mean}	ϵ_{max}	ΔE_{mean}	ΔE_{max}
D65/Object	CRT	-	0.0078	0.1461	0.5477	2.9247
	R-Dupont	F2	0.1651	0.8682	1.1459	4.4970
	R-Object	D65	0.0024	0.0333	0.1813	1.3699
A/Dupont	CRT	-	0.0104	0.2251	0.8099	2.3200
	R-Dupont	F2	0.1467	0.8819	1.0660	3.1110
	R-Object	D65	0.0807	2.1886	0.8314	3.6929
E/Dupont	CRT	-	0.0117	0.2922	0.6538	5.2118
	R-Dupont	F2	0.1932	1.0756	1.0873	4.1710
	R-Object	D65	0.0039	0.1262	0.2082	1.3693
F2/Dupont	CRT	-	0.0273	1.4433	1.1807	4.1951
	R-Dupont	F2	0.0066	0.0524	0.1984	0.7793
	R-Object	D65	0.3180	6.4231	1.5785	3.0799
CRT	CRT	-	≈ 0	≈ 0	≈ 0	≈ 0
	R-Dupont	F2	0.8350	2.9225	1.4452	7.3561
	R-Object	D65	0.8048	17.0150	2.2750	6.6760

Table 3: The error analyses on the three radiance sets are tabulated for a radiometric colorimeter with respect to the expected radiance sets used to compute $E[\mathbf{g}\mathbf{g}^T]$. The original sources for R-Dupont and R-Object are given in the third column. $\gamma = 0$.

using the radiance sets consisting of CRT monitor radiances [1], the Dupont paint set [7] illuminated by the F2 fluorescent, the Object set [7] illuminated by the D65 incandescent. The names of the radiance sets are listed in the second column of Table 3. The table shows that the correlation matrix for radiometric measurements can greatly effect the results of the experiment. Spectra which have unique features can cause problems. Examples of such features include the spectral peaks in F2 illuminated objects and the very low dimensionality of the CRT set (linear combinations of three color guns).

In the last experiment, a combination colorimeter was designed with $\gamma = 0.2$ in Eq(5). A uniform detector, two Gaussian-shaped filters, and two sum-of-Gaussians shaped filters were used. The same five Color Savvy LEDs from the first experiment were used as the internal illuminants. The correlation matrix $E[\mathbf{r}\mathbf{r}^T]$ was obtained from the Dupont reflectance set, and $E[\mathbf{g}\mathbf{g}^T]$ was set equal to the $N \times N$ identity matrix $\mathbf{I}$. The latter assumes maximum ignorance of the data set being measured; the expected $\mathbf{g}$ is independent, identically distributed with $\sigma = 1$.

The optimal color filters for this experiment are shown in Figure 3. The goodness with respect to all measurement spaces is listed in Table 4. The results of the photometric and radiometric experiment are shown in Table 5. Table 6 shows the radiometric measurements for the combination colorimeter with respect to different expected radiance sets used for $E[\mathbf{g}\mathbf{g}^T]$.

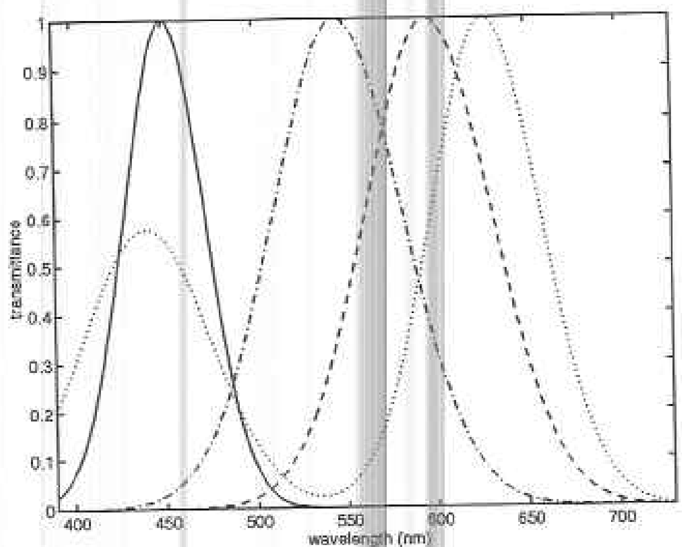

Figure 3: These are the optimal color filter set for the combination colorimeter. $\gamma = 0.2$.

meas.	color matching function	ν	τ
Photometric $\mathbf{V}(\mathbf{M})$	$\mathbf{A}$	0.9966	0.9943
	$\mathbf{A}_A^*$	0.9966	0.9751
	$\mathbf{A}_{D65}$	0.9968	0.9941
	$\mathbf{A}_F$	0.8578	0.5466
Radiometric $\mathbf{W}(\mathbf{M})$	$\mathbf{A}^*$	0.9956	0.9943

Table 4: The goodness of the measurement matrices in the first column are found with respect to the color matching functions in the second. The * denotes the color matching function that the color filters were designed for. The color filters were constrained to two Gaussian and two sum of Gaussians shaped transmittance spectra. $\gamma = 0.2$.

Spec. ⇓	(meas. type, viewing illum.)	ϵ_{mean}	ϵ_{max}	ΔE_{mean}	ΔE_{max}
Munsell	**V**, A	3.0577e-05	6.2071e-04	0.0284	0.1955
Dupont	**V**, A	3.9036e-06	3.9209e-05	0.0157	0.1056
Litho	**V**, A	4.0124e-05	9.3701e-05	0.0515	0.1100
Object	**V**, A	3.0375e-04	0.0051	0.0802	0.4807
Munsell	**V**, D65	5.0555e-04	0.0086	0.0801	0.3589
Dupont	**V**, D65	9.2106e-05	6.6574e-04	0.0750	0.4432
Litho	**V**, D65	0.0042	0.0079	0.3332	0.5660
Object	**V**, D65	4.6063e-04	0.0182	0.1074	0.6099
Munsell	**V**, F2	0.0156	0.2758	0.3313	1.0323
Dupont	**V**, F2	0.0019	0.0313	0.1698	1.0587
Litho	**V**, F2	0.0063	0.0247	0.3017	0.7926
Object	**V**, F2	0.0411	2.3402	0.2981	1.9261
CRT	**W**	0.0020	0.0472	0.6501	2.2641
R-Dupont	**W**, F	0.2335	1.4241	1.4529	4.4123
R-Object	**W**, D65	0.0366	0.4300	0.7558	1.8866

Table 5: A sum of Gaussians optimal color filter set, designed under viewing illuminant A with $\gamma = 0.2$, is tested for its color measurement accuracy using both reflectance sets and radiance sets. The five Color Savvy LEDs were used as the internal illuminant. $E[\mathbf{g}\mathbf{g}^T] = \mathbf{I}$

Radiometric device, $\nu = 0.9975$, $\tau = 0.9956$						
Expected Spec.	Rad. Spec. $\Downarrow$	orig. source	ϵ_{mean}	ϵ_{max}	ΔE_{mean}	ΔE_{max}
D65/Object	CRT	-	0.0035	0.0464	0.7494	3.2877
	R-Dupont	F2	0.3149	1.9202	1.9052	8.4037
	R-Object	D65	0.0070	0.1036	0.3447	1.4068
A/Dupont	CRT	-	0.0284	1.4123	1.2790	7.5217
	R-Dupont	F2	0.1502	0.9492	1.1173	3.7477
	R-Object	D65	0.0782	1.4040	0.9868	2.7033
E/Dupont	CRT	-	0.0041	0.0700	0.7455	2.8641
	R-Dupont	F2	0.3927	2.4746	1.9467	7.9378
	R-Object	D65	0.0089	0.1791	0.3939	1.4984
F2/Dupont	CRT	-	0.0426	2.3552	1.2767	5.5236
	R-Dupont	F2	0.0036	0.0285	0.3298	1.3349
	R-Object	D65	0.3107	3.9038	2.2303	4.7184
CRT	CRT	-	≈ 0	≈ 0	≈ 0	≈ 0
	R-Dupont	F2	35.4696	126.6670	22.2008	111.9266
	R-Object	D65	0.8402	9.4696	4.3865	27.5697

Table 6: The error analysis on the three radiance sets are tabulated for a radiometric colorimeter. The original sources for R-Dupont and R-Object are given in the third column. $\gamma = 0.2$, and the sum of Gaussians optimal color filter set was used.

4 Summary and Conclusions

The results of three design experiments; the photometric colorimeter, the radiometric colorimeter, and the combination colorimeter are described. The colorimeter designed for only photometric and radiometric measurements outperformed the combination colorimeter as expected. However, the use of the measure of goodness and the weighting factor allowed for a combination design that makes reasonable photometric and radiometric measurements. Radiometric measurements are sensitive to the correlation matrix used in the linear minimum mean squared estimation of tristimulus values. Choosing a nonspecialized data set for the estimate of the correlation is recommended.

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APPENDIX A. Vora's measure of goodness [6]

The measure of goodness, or how well the columns of matrix **B** spans another matrix **A**, is given by

$$\nu(\mathbf{A},\mathbf{B}) = \frac{[\sum_{i=1}^{\alpha} \lambda_i^2(\mathbf{O}^T\mathbf{N})]}{\alpha} = \frac{Trace(\mathbf{N}^T\mathbf{O}\mathbf{O}^T\mathbf{N})}{\alpha}, \tag{8}$$

where $\mathbf{O}$ is an matrix of orthonormal basis vectors that satisfies

$$R(\mathbf{O}) = R(\mathbf{P_B}) = R(\mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T), \tag{9}$$

$\mathbf{N}$ is an matrix of orthonormal basis vectors that satifies

$$R(\mathbf{N}) = R(\mathbf{P_A}) = R(\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T), \tag{10}$$

and α is the number of singular values in the SVD of $\mathbf{O}^T\mathbf{N}$. Thus for the photometric measurement matrix $\mathbf{V}$, the space that needs to be spanned is $\mathbf{A_L}$, the color matching matrix under illuminant $\mathbf{l}$; the photometric measure of goodness is given by $\nu(\mathbf{A_L}, \mathbf{V})$. Similarly, the radiometric measure of goodness is given by $\nu(\mathbf{A}, \mathbf{W})$.

APPENDIX B. Color quality factor

The Color Savvy company uses a color quality factor to estimate the performance of a filter set. This measure will be called the τ measure. If the color filters are to be used under the L illuminant, then $\mathbf{A}_L = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$is the matrix of appropriate color matching functions. The color filter matrix, $\mathbf{M}$, should be a linear transformation from $\mathbf{A}_L$, or

$$\mathbf{M}\mathbf{B}^{(i)} \approx \mathbf{a}_i \tag{11}$$

where $\mathbf{a}_i$ is one column of the Nx3 color matching function and $\mathbf{B}^{(i)}$ is the optimal linear transformation. The linear minimum mean square estimate of $\mathbf{B}^{(i)}$ can be found using the error function

$$\min_{w.r.t.\mathbf{B}^{(i)}} \quad ([\mathbf{B}^{(i)}]^T\mathbf{M}^T - \mathbf{a_i}^T)(\mathbf{M}\mathbf{B}^{(i)} - \mathbf{a_i}) \tag{12}$$

which is expanded to

$$\min_{w.r.t.\mathbf{B}^{(i)}} \quad [\mathbf{B}^{(i)}]^T(\mathbf{M}^T\mathbf{M})\mathbf{B}^{(i)} - 2[\mathbf{B}^{(i)}]^T\mathbf{M}^T\mathbf{a_i} + \mathbf{a_i}^T\mathbf{a_i} \tag{13}$$

where $i = 1$, 2 or 3. By taking the matrix derivative with respect to $\mathbf{B}^{(i)}$ and set equal to $\mathbf{0}$, it can be shown that $2(\mathbf{M}^T\mathbf{M})\mathbf{B}^{(i)} - 2\mathbf{M}^T\mathbf{a_i} = \mathbf{0}$, or equivalently

$$\mathbf{B}^{(i)} = (\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T\mathbf{a_i}. \tag{14}$$

The τ measure of goodness is

$$\tau(\mathbf{A}_L, \mathbf{M}) = \min_i \quad \{\frac{||\mathbf{a_i}^T\mathbf{M}\mathbf{B}^{(i)}||}{||\mathbf{a_i}||^2}\}. \tag{15}$$

Placing Eq(14) into Eq(15),

$$\tau(\mathbf{A}_L, \mathbf{M}) = \min_i \quad \{\frac{||\mathbf{a_i}^T\mathbf{M}(\mathbf{M}^T\mathbf{M})^{-1}\mathbf{M}^T\mathbf{a_i}||}{||\mathbf{a_i}||^2}\}, \tag{16}$$

or

$$\tau(\mathbf{A}_L, \mathbf{M}) = \min_i \quad \{\frac{||\mathbf{P}_M\mathbf{a_i}||^2}{||\mathbf{a_i}||^2}\}, \tag{17}$$

Eq(17) shows that the τ measure is an extension of Neugebauer's q-factor.